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UNSTEADY BEHAVIOR OF AN ELASTIC BEAM FLOATING ON THE SURFACE OF AN INFINITELY DEEP FLUID

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The effect of initial disturbances and unsteady external loading on an elastic beam of finite length which floats freely on the surface of an ideal incompressible fluid is studied in a linear treatment. The fluid flow is considered potential. The beam deflection is sought in the form of an expansion in the eigenfunctions of beam vibrations in vacuum with time-dependent amplitudes. The problem reduces to solving an infinite system of integrodifferential equations for unknown amplitudes. The memory functions entering this system are determined by solving the radiation problem. The beam behavior is studied for various loads with and without allowance for the weight of the fluid. The effect of fluid depth on beam deformation was determined by comparing with the previously obtained solutions of the unsteady problem for a beam floating in shallow water.

Key words: *floating elastic plate, infinitely deep fluid, unsteady external loading.*

Introduction. Recent increased interest in the behavior of large floating structures has been motivated by the design of platforms for various purposes [1]. In mathematical modeling, such platforms are often treated as thin elastic plates. The problem of the unsteady behavior of a finite elastic plate floating on a free fluid surface has been studied insufficiently even in a linear approximation. The behavior of a beam plate floating in shallow water [2, 3] and on the surfaces of a finite-depth fluid [4] has been studied in the two-dimensional case. For the three-dimensional case, methods have been developed to solve the unsteady hydroelastic problem for a circular plate in shallow water [5] and for a rectangular plate floating on the surface of an infinitely deep fluid [6, 7].

In the present work, the mode-expansion method used in [6, 7] is employed to solve the two-dimensional problem of the unsteady behavior of an elastic beam floating on the surface of an infinitely deep fluid. In the two-dimensional case, it is possible to avoid the assumptions introduced in [6, 7] to determine the memory functions because the behavior of the damping coefficients in the high-frequency limit is known exactly for the radiation problem. The attempt in [8] to solve the examined problem was not been brought to numerical calculations.

The present study largely uses the results of [9]; therefore, the common fragments will be frequently omitted and indicated by referring to that work.

1. Formulation of the Problem. Let an elastic homogeneous beam of length $2L$ float freely on the surface of an ideal incompressible fluid. The fluid surface that is not covered by the beam is free, and the fluid depth is infinite. The fluid flow is considered potential. The coordinate origin corresponds to the middle of the beam. The beam draft is ignored. The velocity potential $\phi(x, y, t)$ in the fluid satisfies the Laplace equation

$$\Delta\phi = 0 \qquad (|x| < \infty, \quad y < 0),$$

where x is the horizontal coordinate, y is the vertical coordinate directed upward, and t is time.

At $y = 0$, the dynamic and kinematic conditions become

$$\frac{p}{\rho} = -\frac{\partial\phi}{\partial t} - gw, \qquad \frac{\partial\phi}{\partial y} = \frac{\partial w}{\partial t}, \tag{1.1}$$

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where $p(x,y,t)$ is the fluid pressure, $w(x,t)$ is the elevation of the free surface or the normal deflection of the beam, ρ is the water density, and g is the gravitational acceleration. On the free surface of the fluid,

$$p = 0 \qquad (|x| > L, \quad y = 0). \tag{1.2}$$

Away from the beam,

$$|\nabla\phi| \to 0 \qquad (x^2 + y^2 \to \infty). \tag{1.3}$$

The normal deflection of an Euler beam is described by the equation

$$D\,\frac{\partial^4 w}{\partial x^4} + \rho_1 h_1\,\frac{\partial^2 w}{\partial t^2} - p(x,0,t) = -p_e(x,t) \qquad (|x| \leqslant L), \tag{1.4}$$

where D, ρ_1, and h_1 are the cylindrical rigidity, density, and thickness of the beam, respectively. The function $p_e(x,t)$ is specified and describes the external load on the beam.

The free-edge conditions are specified at the beam edges, i.e., the bending moment and the shear forces are set equal to zero:

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial x^3} = 0 \qquad (|x| = L).$$

At the initial time, the following functions are given:

$$\phi_0(x,y) = \phi\Big|_{t=0}, \quad w_0(x) = w\Big|_{t=0}, \quad w_1(x) = \frac{\partial w}{\partial t}\Big|_{t=0}.$$

They are not independent and should satisfy boundary conditions (1.1)–(1.3).

It is also of interest to solve this problem under the assumption that the fluid is weightless. This model is used in impact theory to study the short-term effect on a floating elastic body [8]. For the case of a weightless fluid, condition (1.2) on the free surface is replaced by

$$\phi = 0 \qquad (|x| > L, \quad y = 0).$$

Let us convert to dimensionless variables (denoted by an asterisk):

$$(x^*, y^*, w^*) = \frac{1}{L}\,(x,y,w), \quad t^* = t\sqrt{\frac{g}{L}}, \quad \phi^* = \frac{\phi}{\sqrt{gL^3}}, \quad (p, p_e) = \frac{1}{\rho g L}\,(p, p_e).$$

Below, the following dimensionless coefficients are used:

$$\gamma = \frac{D}{\rho g L^4}, \qquad \chi = \frac{\rho_1 h_1}{\rho L}.$$

The beam deflection is sought in the form of an expansion in the eigenfunctions of vibrations of a free-ends beam in vacuum (below the asterisks are omitted)

$$w(x,t) = \sum_{n=0}^{\infty} X_n(t) W_n(x), \tag{1.5}$$

where the functions $X_n(t)$ are to be determined and the functions $W_n(x)$ are solutions of the following spectral problem:

$$W_n^{(IV)} = \lambda_n^4 W_n \qquad (|x| \leqslant 1),$$

$$W'_{2k} = W_{2k+1} = 0 \quad (x = 0), \qquad W''_n = W'''_n = 0 \quad (|x| = 1).$$

The prime denotes differentiation with respect to x. The values of the functions $W_n(x)$ are given in [9].

2. Equations of Motion. According to [6, 7], for the unknown eigenfunctions $X_n(t)$ we have the infinite system of integrodifferential equations

$$\sum_{m=0}^{\infty}\Big[(\chi\delta_{mn} + \bar{a}_{mn})\ddot{X}_m + \int_0^t \dot{X}_m(\tau) K_{mn}(t-\tau)\,d\tau\Big] + (1 + \gamma\lambda_n^4) X_n = -F_n(t) \tag{2.1}$$

$$(n = 0, 1, 2, \ldots)$$

with the initial conditions

$$X_n(0) = x_n^0, \qquad \dot{X}_n(0) = x_n^1,$$

where

$$\bar{a}_{mn} = \lim_{\omega\to\infty} a_{mn}(\omega), \quad F_n(t) = \int_{-1}^{1} p_e(x,t) W_n(x)\,dx, \quad K_{mn}(t) = \frac{2}{\pi}\int_0^\infty b_{mn}(\omega)\cos\omega t\,d\omega,$$

$$x_n^0 = \int_{-1}^{1} w_0(x) W_n(x)\,dx, \qquad x_n^1 = \int_{-1}^{1} w_1(x) W_n(x)\,dx, \tag{2.2}$$

and δ_{mn} is the Kronecker symbol. The overdot denotes differentiation with respect to time. To obtain Eq. (2.1), we substitute expansion (1.5) into (1.4), multiply the result by $W_m(x)$, and integrate the resulting equation over x from -1 to 1. The functions $a_{mn}(\omega)$ and $b_{mn}(\omega)$ are determined by solving the radiation problem and, by analogy with the seakeeping problem for the ship, they are called the added-mass and damping coefficients, respectively.

The radiation problem is equivalent to determining the behavior of the fluid for specified harmonic vibrations of the vertical velocity with frequency ω on the segment of the upper boundary at $|x| \leqslant 1$. The remaining part of the upper boundary of the fluid is a free surface. The vibrational motion of the fluid is considered steady-state and all the functions determining the motion characteristics are proportional to $\exp(i\omega t)$. The solution of this problem is presented in [9].

In the limiting cases of low and high vibration frequencies, the behavior of the functions $a_{mn}(\omega)$ and $b_{mn}(\omega)$ is well known. Explicit expressions for $\bar{a}_{mn}$ are given in [9].

The damping coefficients $b_{mn}(\omega)$ are equal to zero for $\omega = 0$ and tend to zero as $\omega \to \infty$. In the case of high frequencies, the asymptotic expression for these coefficients is written as

$$b_{mn}(\omega) = c_{mn}/\omega + O(\omega^{-2}) \qquad (\omega \to \infty),$$

where $c_{mn} = \pi U_n U_m$. The values of U_n are given in [9].

In the calculation of the so-called memory function $K_{mn}(t)$ in (2.2), the damping coefficients for rather high frequencies are approximated by the function

$$b_{mn}(\omega) = c_{mn}/\omega \qquad (\omega \geqslant \omega_*). \tag{2.3}$$

As shown in [9], the value of ω_* should not be smaller than 30.

Substituting (2.3) into (2.2), we obtain

$$K_{mn}(t) = \frac{2}{\pi}\Big[\int_0^{\omega_*} b_{mn}(\omega)\cos\omega t\,d\omega - c_{mn}\mathrm{Ci}(\omega_* t)\Big], \tag{2.4}$$

where $\mathrm{Ci}(\,\cdot\,)$ is the integral cosine. For small values of t, the function $K_{mn}(t)$ has the logarithmic singularity

$$K_{mn}(t) \to -c_{mn}\ln t \qquad (t \to 0).$$

3. Numerical Results. Using the reduction method, we replace the infinite series in expansion (1.5) by the sum with the number of terms $2(N+1)$. In view of the evenness properties for the functions $W_n(x)$, the system of integrodifferential equations (2.1) splits into two separate systems for even and odd numbers n. This system is solved by the method of central finite differences with a constant time step.

The first term (2.4) on the left side is found numerically using the values of $b_{mn}(\omega)$ determined for a discrete set of frequencies. In this case, the frequency range 0–ω_* is divided into segments with a nonuniform step (for more details see [9]). Within each segment, the value of $b_{mn}(\omega)$ is also determined at the middle point, which allows one to introduce a quadratic approximation for this function and perform the analytical integration in (2.4) on each segment (for more details see [6, 7]). The convolution type integral in (2.1) is calculated using the method described earlier [5–7]. As a result, for the time $t = k\zeta$, where ζ is the time step, we obtain two uncoupled systems of linear algebraic equations

$$B_1 \boldsymbol{A}_1^{k+1} = \boldsymbol{C}_1^k, \qquad B_2 \boldsymbol{A}_2^{k+1} = \boldsymbol{C}_2^k, \tag{3.1}$$

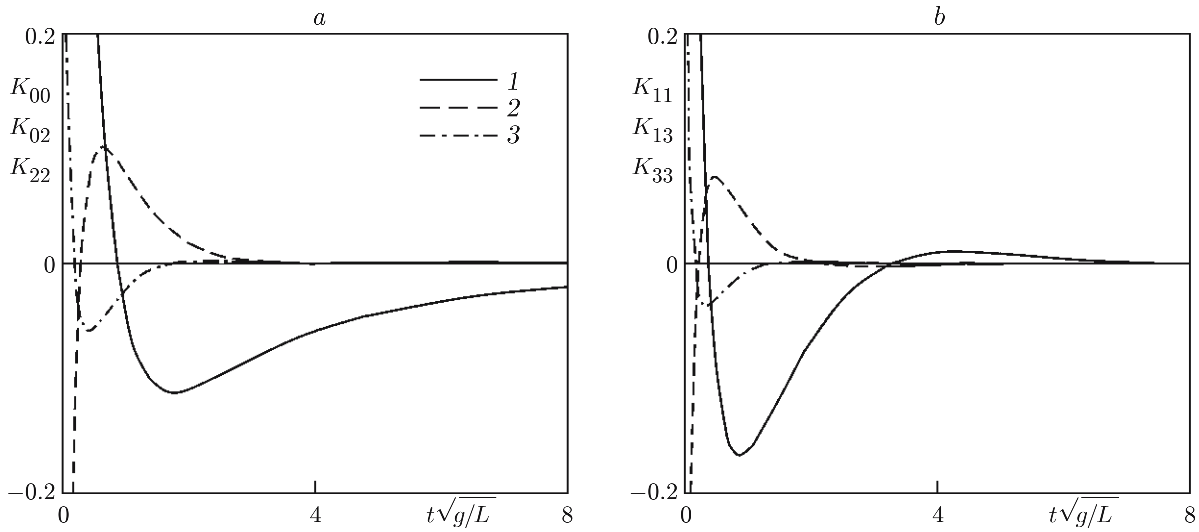

Fig. 1. Time dependences of the memory functions K_{nm}: (a) K_{00} (1), K_{02} (2), and K_{22} (3); (b) K_{11} (1), K_{13} (2), and K_{33} (3).

where the square matrices B_1 and B_2 are completely filled and do not depend on the time step and the vectors $\boldsymbol{A}_1^m$ and $\boldsymbol{A}_2^m$ are

$$\boldsymbol{A}_1^m = \{X_0(m\zeta), X_2(m\zeta), \ldots, X_{2N}(m\zeta)\}^{\rm t}, \quad \boldsymbol{A}_2^m = \{X_1(m\zeta), X_3(m\zeta), \ldots, X_{2N+1}(m\zeta)\}^{\rm t}.$$

In addition to the right side of Eqs. (2.1), the vectors $\boldsymbol{C}_1^k$ and $\boldsymbol{C}_2^k$ contain terms that take into account the values of $X_n(t)$ in the previous steps. The superscript "t" denotes transposition.

The proposed method is universal in the sense that the values of the added-mass and damping coefficients, and, hence, the memory functions $K_{mn}(t)$ do not depend on the particular properties of the plate and the type of its unsteady motion. The matrices B_1 and B_2 in (3.1) depend on the properties of the plate but do not depend on the type of external perturbation.

To obtain the solution corresponding to the case of a weightless fluid, the memory functions $K_{mn}(t)$ in (2.1) are set equal to zero and the problem is reduced to solving a system of ordinary differential equations.

Curves of the memory functions versus time for the first modes are presented in Fig. 1. It should be noted that the most significant of these functions are those which correspond to the solid-state vibrations of the beam. For the elastic modes, the memory function are different from zero only for rather small times.

Next, we consider the beam behavior caused by its initial deformation in the fluid at rest. Examples of numerical calculations for a beam floating in shallow water are given in [3] and for a beam floating on a surface of a finite-depth fluid in [4]. We used the following initial conditions:

$$w_0(x) = a \exp(-50x^2/(7L^2)), \qquad \phi_0 = w_1 = 0.$$

Here, in the dimensional variables, the multiplier a has the dimension of length. There is no external loading, and hence, in (1.4), we have $p_e = 0$. The calculations were performed for $\gamma = 0.0032$ and $\chi = 0$. In all calculations given below, $N = 6$, and a further increase in N has little effect on the result. System (2.1) was integrated with a dimensionless time step of 0.01. In this problem, only the even modes are taken into account since the initial deformation of the beam is symmetric about the coordinate origin. Curves of the normal deflections of the beam versus time are presented in Fig. 2. For $t < \sqrt{L/g}$, wave processes in the heavy fluid have no time to develop and the beam behavior is similar in both cases. With time, however, the difference becomes significant. For the beam floating on the surface of a heavy fluid, the initial deformation disappears with time and the beam takes a horizontal unperturbed position. The vibrations attenuate more rapidly in the middle part of the beam than at its ends. For the beam in a weightless fluid, the deflections both at the center and at the edges do not decrease with time since in this case there are no surface-wave generation and energy dissipation.

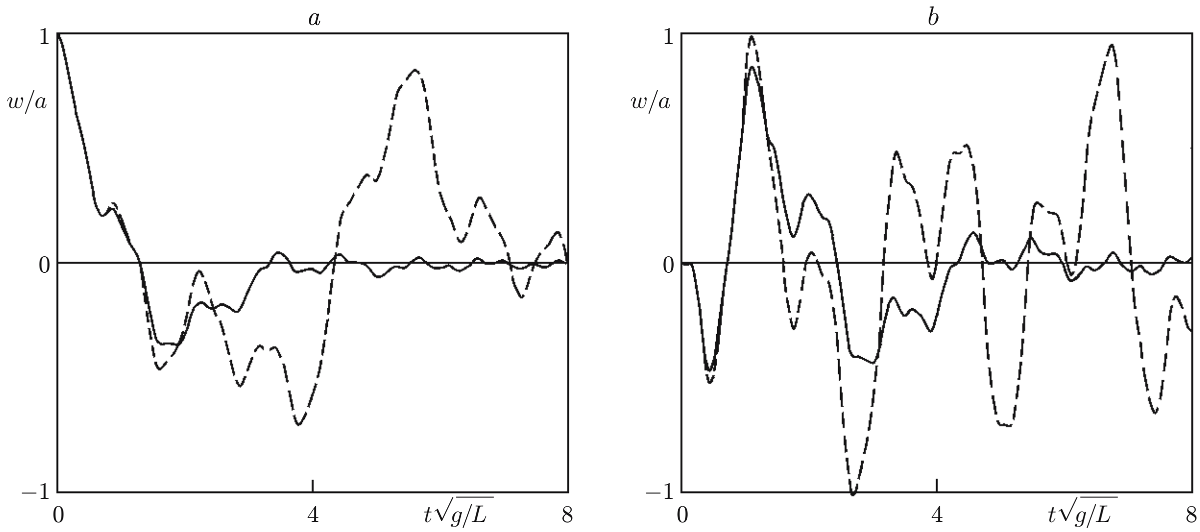

Fig. 2. Beam deflections due to its initial deformation versus time: (a) $x = 0$; (b) $x = L$; the solid curves refer to a heavy fluid; the dashed curves refer to a weightless fluid.

Unsteady external action on the beam was studied using two types of loading considered earlier [2] for a beam floating in shallow water. For impact loading, the external-pressure distribution is specified in the form

$$p_e(x,t) = a\rho g F(x) B(t), \tag{3.2}$$

where the constant a has the dimension of length,

$$F(x) = \begin{cases} 1-(x-l)^2/s^2, & |x-l| \leqslant s, \\ 0, & |x-l| > s, \end{cases} \qquad |l| + s < L, \tag{3.3}$$

$$B(t) = \begin{cases} t/b, & t \leqslant b; \\ 2 - t/b, & b \leqslant t \leqslant 2b; \\ 0, & t > 2b. \end{cases}$$

At the initial time, the fluid and the beam are at rest.

The initial parameters are as follows: $D = 4.476 \cdot 10^{10}$ kg · m^2/sec^2, $\rho = 10^3$ kg/m^3, $\rho_1 h_1/\rho = 1$ m, $L = 200$ m, $s = 40$ m, and $b = 0.5$ sec.

The calculations for a beam of finite dimensions are compared with the well-known solution for an infinite beam floating on the surface of a deep heavy fluid (see, for example, [10]). For loading in the form of (3.2), this solution is written as

$$w(x,t) = \frac{a\rho g}{\pi b} \int_0^\infty \frac{Y(k,t)\tilde{F}(k)\cos kx}{Dk^4 + \rho g}\, dk,$$

where

$$Y(k,t) = \begin{cases} \Omega^{-1}\sin\Omega t - t, & 0 \leqslant t \leqslant b; \\ \Omega^{-1}[2\sin\Omega(b-t) + \sin\Omega t] - 2b + t, & b \leqslant t \leqslant 2b; \\ \Omega^{-1}[2\sin\Omega(b-t) - \sin\Omega(2b-t) + \sin\Omega t], & t > 2b; \end{cases}$$

$$\Omega^2(k) = k(Dk^4 + \rho g)/(\rho + \rho_1 h_1 k).$$

$\tilde{F}(k)$ is the Fourier transform of the function $F(x)$ in (3.3):

$$\tilde{F}(k) = \frac{4}{sk^2}\Big(\frac{\sin ks}{ks} - \cos ks\Big).$$

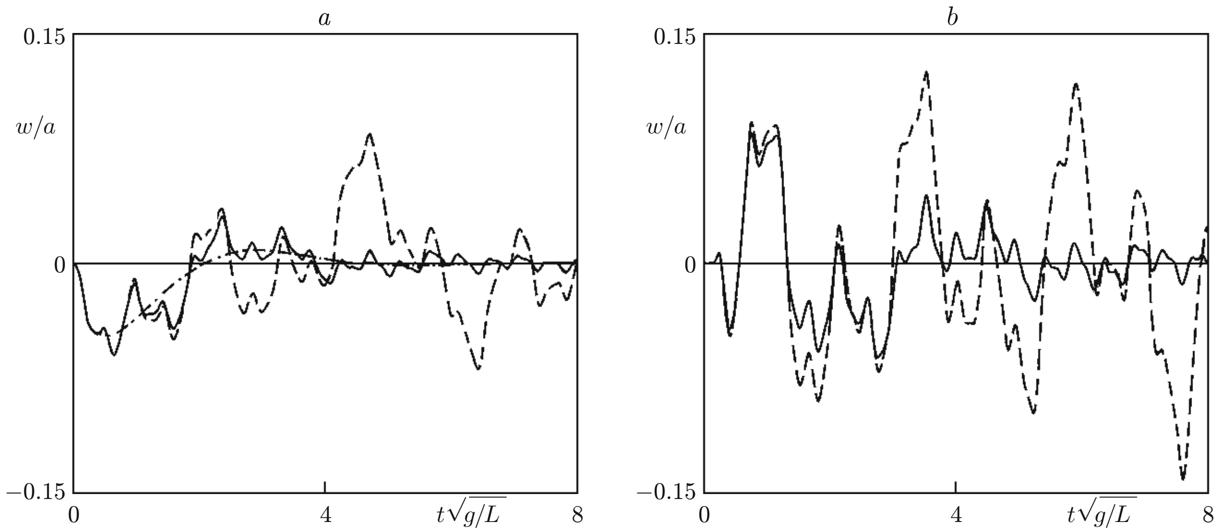

Fig. 3. Time dependences of beam deflections due to symmetric impact loading ($l = 0$): (a) $x = 0$; (b) $x = L$; the solid curves refer a heavy fluid, the dashed curves refer to a weightless fluid, and the dot-and-dashed curve refers to an infinite beam.

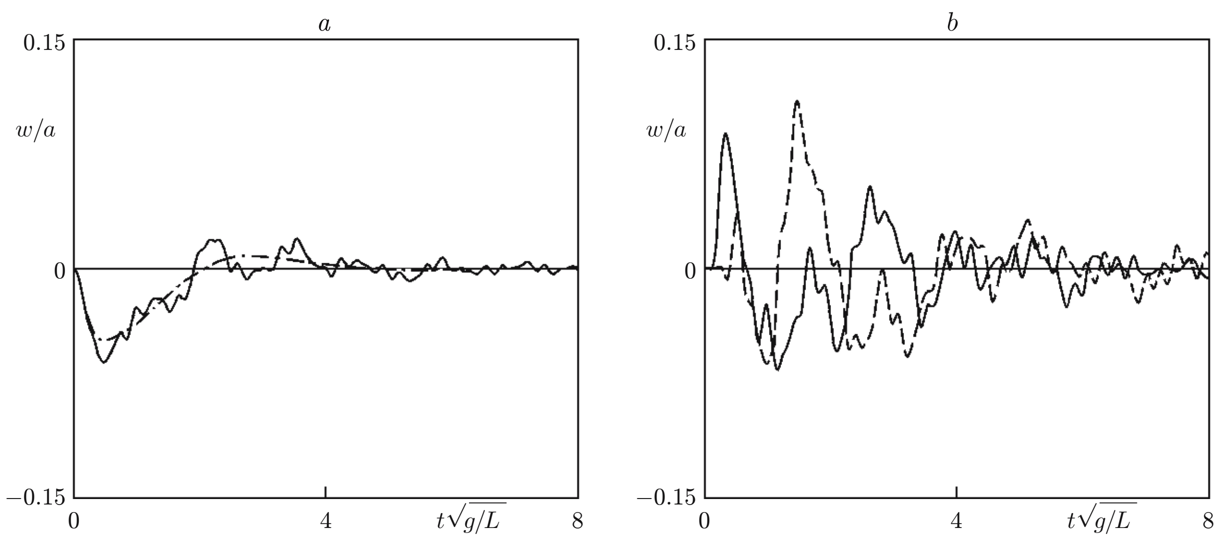

Fig. 4. Time dependences of beam deflections due to asymmetric impact loading ($l = 0.5L$): (a) $x = l$ (the solid curve refers to a finite beam; the dot-and-dashed curve refers to an infinite beam); (b) $x = L$ (solid curve) and $x = -L$ (dashed curve).

Figure 3 gives calculation results for the case of symmetric external loading. It should be noted that at the pressure epicenter in the time interval where the pressure acts ($t\sqrt{g/L} \leqslant 2b\sqrt{g/L} \approx 0.22$), all three solutions coincide but after the termination of the external action, the beam deflections differ significantly in different cases. For the beam floating on the surface of a heavy fluid, the initial state is established with time, and for the beam in a weightless fluid, the deflections both at the pressure epicentre and at the edges do not decrease with time. As for a beam floating in shallow water [2], in the case of an infinitely deep fluid, the beam vibrations are larger at the edges than at the pressure epicenter. The maximum deflection of the edge achieved at $t\sqrt{g/L} \approx 0.73$ is almost 1.4 times the maximum deflection at the pressure epicentre.

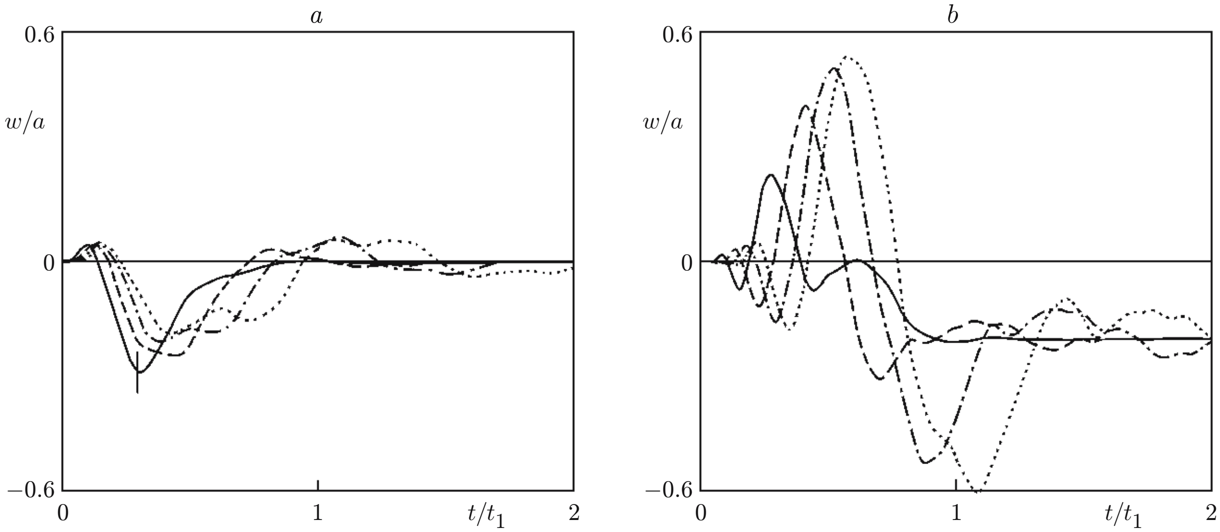

Fig. 5. Time dependences of the beam deflections due to a moving load at $x = 0$ (a) and $x = -L$ (b); the solid, dashed, dot-and-dashed, and dotted curves correspond to initial velocities of motion of the load $v/\sqrt{gL} = 0.3$, 0.5, 0.7, and 0.9, respectively.

The calculation results for the case of asymmetric pressure application ($l = 0.5L$) are presented in Fig. 4, which shows the normal deflections of the beam at the pressure epicentre at $x = l$ (the solid and dot-and-dashed curves in Fig. 4a for finite and infinite beams, respectively) and the vibrations of the edges at $x = L$ (the solid curve) and $x = -L$ (the dashed curve) (Fig. 4b). The maximum deflections at the pressure epicentre and at the right edge of the beam practically coincide with the corresponding values in the symmetric case (compare Figs. 3 and 4), and the maximum deflection of the left edge is somewhat larger. The maximum deflections occur at the edges at different times. The maximum deflection is first (at $t\sqrt{g/L} \approx 0.32$) reached at the beam end the nearest to the pressure region and then (at $t\sqrt{g/L} \approx 1.46$) at the opposite end.

The effect of a moving load was studied by modeling the landing of an airplane. It is assumed that at the initial time, the beam and the fluid are at rest and the load smoothly, at a velocity v, touches the beam in the region with center at the point $x = x_0$ and then performs uniformly decelerated motion to the left until the full stop at the point $x = x_1$ at the time $t = t_1 \equiv 2(x_0 - x_1)/v$. The external-load distribution is specified in the form

$$p_e(x,t) = a\rho g[1 - \exp(-bt)]F(x,t),$$

where the function $F(x,t)$ is defined by relation (3.3) with the dependence

$$l(t) = \begin{cases} x_0 - vt + v^2t^2/[4(x_0 - x_1)], & 0 \leqslant t \leqslant t_1; \\ x_1, & t > t_1. \end{cases}$$

The former initial parameters are used but in this case, $s = 0.1L$, $x_0 = 0.7L$, $x_1 = -0.7L$, and $b = 20/t_1$.

It is known [10] that there is the critical velocity of flexural-gravity waves $v_{\rm cr}$; for an infinitely deep fluid, it is defined by the relation

$$v_{\rm cr} = 2(Dg^3/(27\rho))^{1/8}.$$

For the specified parameters, this corresponds to the dimensionless value of $v_{\rm cr}/\sqrt{gL} \approx 0.64$. We note that the critical velocity of flexural-gravity waves in an infinitely deep fluid is almost twice the critical velocity of these waves in shallow water.

Figure 5 gives curves of the normal deflections of the beam at the points $x = 0$ and $x = -L$ versus time for various initial velocities of motion of the load. In Fig. 5a, the vertical line shows the moment when the load peak intersects the coordinate origin $t/t_1 \approx 0.29$. As in the case of shallow water, the maximum deflection of the beam is reached at this moment only if the load moves rather slowly. As v increases, the maximum deflection at $x = 0$ decreases and its occurrence is observed after the load peak passes through the coordinate origin. At $t > t_1$, the beam vibrations calm down and gradually take values that correspond to a distributed static load with center at the point $x = x_1$. The solution of this steady-state problem is easy to obtain from system (2.1), which in this case reduces to a simple system of linear algebraic equations. The static deflections do not depend on the fluid depth and are given in [2]. As in the case of shallow water, the vibrations of the right end are much smaller than those at the examined points. In Fig. 5 it is evident that the largest deflections, as a rule, arise at the left end of the beam. An increase in the initial loading velocity leads to more intense vibrations of the left edge of the beam and the larger time of establishment of the static regime compared to the time of motion of the load.

Conclusions. The results presented above, in aggregate with those obtained earlier [2], show the effect of the fluid depth on the behavior of a floating elastic beam subjected to initial perturbations and unsteady external loading. The comparison of the maximum deflections of the beam under the action of the same loads for shallow water and infinitely deep fluid, one can see that the beam vibrations in the latter case are larger as a rule. The proposed calculation method is universal and can be extended to the case of finite-depth fluids.

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